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A NONLINEAR REGULARIZING ALGORITHM FOR SOLVING ONE CLASS OF INVERSE PROBLEMS OF HEAT CONDUCTION

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A nonlinear regularizing algorithm is proposed for solving ill-conditioned systems of equations, which takes account of typical a priori information about the solution sought.

Numerous formulations of the inverse heat conduction problem [1] (specifically, a parametrized identification of heat conduction processes) lead to the system of linear algebraic equations of the form

$$K\varphi = f \tag{1}$$

As a rule, system (1) is ill-conditioned or degenerate and in order to construct a stable (regularized) solution, different methods of regularization of the solution are used [2, 3]. In this work, a method is presented for constructing a regularized solution on the basis of the singular expansion of the matrix K, taking into account preliminary information that is typical for the considered problem.

Linear Regularizing Algorithm. We assume for definiteness that the matrix K is of order $N_f \times N_\varphi$, where φ and f are vectors of appropriate dimensionality. The representation [4] $K = U\Lambda V^T$ is called a singular expansion of the matrix K, in which U and V are orthogonal matrices of order $N_f \times N_f$, $N_\varphi \times N_\varphi$, where T is the transpose sign, Λ is a matrix of order $N_f \times N_\varphi$ with elements

$$\{\Lambda\}_{i,j} = \begin{cases} \lambda_i, & i = j; \\ 0, & i \neq j. \end{cases}$$

The values $\lambda_i \geq 0$, $i = 1, 2, \ldots, N_\varphi$, are called singular numbers of the matrix K. Suppose that a) $N_f \geq N_\varphi$; b) singular numbers are ordered: $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{N_\varphi} \geq 0$; c) instead of the exact right-hand side of f, the vector $\tilde{f} = f + \eta$, where η is a random vector with zero mean, reflecting errors in specifying the right-hand side of Eq. (1), is specified.

A solution φ_α stable with respect to noise η and to the errors of the computational process realizing a singular expansion can be represented in the following form

$$\varphi_\alpha = Vx_\alpha,$$
$$x_\alpha(j) = r_\alpha(j)\tilde{y}(j), \quad r_\alpha(j) = 1/(\lambda_j + \alpha m(\lambda_j)), \quad 1 \leqslant j \leqslant N_\varphi, \tag{2}$$

where x_α is an N_φ-dimensional vector; $x_\alpha(j)$ is its j projection; $\tilde{y}(j)$ is the j-projection of the vector $\tilde{y} = U^T f$; α is a parameter of regularization; $m(\lambda)$ is a nonincreasing positive function (for example, $m(\lambda) = \lambda^{-\theta}$, $\theta \geq 1$). It can be shown that for an appropriate choice of α the solution φ_α is regularized, i.e., when the errors tend to zero, φ_α converges to the exact pseudosolution of system (1). By not considering the choice of α, we only note that existing algorithms for estimating an optimal (in the sense of a root-mean-square

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error of the solution) value of the parameter of regularization [3, 5] are realized effectively with the use of the singular expansion of the matrix K. Later on, the parameter α is considered to be chosen. We consider two characteristics of a linear regularizing algorithm. The following chain of equalities holds:

$$N_\varphi = \sum_{j=1}^{N_\varphi} \frac{\lambda_j + \alpha m(\lambda_j)}{\lambda_j + \alpha m(\lambda_j)} = \sum_{j=1}^{N_\varphi} \frac{\lambda_j}{\lambda_j + \alpha m(\lambda_j)} + \sum_{j=1}^{N_\varphi} \frac{\alpha m(\lambda_j)}{\lambda_j + \alpha m(\lambda_j)} = M_{de}(\alpha) + M_{co}(\alpha). \quad (3)$$

The first term can be interpreted as the total number of projection x(j) of the vector $x = V^T\varphi$, not distorting the regularizing algorithm (2), and therefore, we call $M_{de}(\alpha)$ the number of degrees of freedom of the regularizing algorithm. The value $M_{co}(\alpha)$ shows the total number of projections x(j), which are practically excluded (due to their low information provision, a small value of the signal/noise ratio = $\lambda_j^2 x^2(j)/\sigma_\eta^2$) from the solution φ_α (transfer to a systematic error b_α [3, p. 143]). These projections can be determined, however, not from the right-hand part of system (1) but from additional a priori information on the vector φ.

A priori Restrictions and Their Inclusion in the Construction of the Solution. Reliable a priori information about the vector φ determines a priori restrictions imposed on the constructed solution φ_α. The following restrictions are typical for the given problem:

1) nonzero projections $\varphi(j)$ for $j \in J_\varphi$ only determine an admissible set Φ_1 for the vector φ:

$$\Phi_1 = \{\varphi : \varphi(j) = 0, \quad j \notin J_\varphi\}, \quad (4)$$

2) the fact that projections $\varphi(j)$ belong to the interval $[\varphi_{min}(j), \varphi_{max}(j)]$ determines the admissible set Φ_2:

$$\Phi_2 = \{\varphi : \varphi_{min}(j) \leqslant \varphi(j) \leqslant \varphi_{max}(j)\}. \quad (5)$$

The sets Φ_1 and Φ_2 are convex and closed.

Evidently, the introduction of a priori restrictions (4), (5) in a linear algorithm (2) by means of a stabilizing factor $m(\lambda)$ is not possible. Therefore, in the present work, in order to take account of (4), (5), we propose using the so-called "method of projections into convex sets" [6, 7]. We will consider the restrictions of type 1), 2) although, in general, there are other restrictions, specific for a particular problem, which determine additional sets Φ_i. We assume that a priori restrictions determine m admissible sets Φ_1, Φ_2, ..., Φ_m and that all these sets are convex and closed.

Construction of a solution that satisfies a priori restrictions has a simple geometrical interpretation: The solution has to belong to the intersection $\Phi_0 = \prod_{i=1}^{m} \Phi_i$ of the admissible sets Φ_i, and each newly introduced restriction narrows the region of admissible solutions. The determination of the element φ^* of the convex set Φ_0 is reduced to the construction of the projection operator into the set Φ_0. The structure of the set Φ_0 can be such that a direct construction of the projection operator into Φ_0 is a complex problem, while the projection operator P_i into any i set from Φ_i has a sufficiently simple form. Consequently, in order to find φ^* it is advisable to use the iteration procedure $\varphi^{(k)} = P^k\varphi^{(0)}$, based on the following statement [7]: If $P = P_m P_{m-1} \ldots P_1$, then a stationary point φ^* of the operator P belongs to Φ_0, and the sequence $\{\varphi^{(k)}\}$ converges to φ^* for any $\varphi^{(0)}$. In order to improve the convergence we introduce operators $T_i = I + \mu_i(P_i - I)$, where I is the unit operator and μ_i is the relaxation parameter. For the operator $T = T_m T_{m-1} \ldots T_1$ the following statement holds [7]: If Φ_0 is not empty and μ_i satisfies the condition

$$0 < \mu_i < 2, \quad 1 \leqslant i \leqslant m, \quad (6)$$

then the sequence

$$\varphi^{(k)} = T^k \varphi^{(0)} \quad (7)$$

converges to φ^* for any initial element $\varphi^{(0)}$. Evidently, for $\mu_i = 1$, i = 1, 2, ..., m, the equality T = P holds.

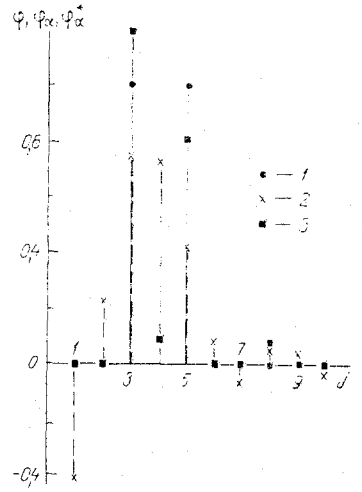

Fig. 1. Results of the computational experiment: 1) vector of exact solution; 2) vector of regularized solution φ_α; 3) vector of regularized solution φ_α^*.

A Nonlinear Algorithm for Constructing a Regularized Solution. In the proposed scheme of nonlinear regularization, in order to save informative components of the solution φ_α we introduce the admissible set Φ_3, defined as

$$\Phi_3 = \{\varphi : \{V^T\varphi\}_j = x_\alpha(j), \quad 1 \leqslant j \leqslant M_{de}(\alpha)\},$$

where $\{V^T\varphi\}_j$ is the j-th projection of the vector $V^T\varphi$; $M_{de}(\alpha)$ is the number of degrees of freedom for the regularizing algorithm defined from (3). Evidently, projections with numbers $M_{de}(\alpha) + 1 \leq j \leq N_\varphi$ are formed basically by the a priori restrictions (4), (5). We note that the set Φ_3 is also convex and closed. The algorithmic realization of the procedure (7) for the sets Φ_1, Φ_2, and Φ_3 is of the form:

Step 0: evaluation of the required solution φ_α and construction of the initial approximation

$$\varphi_\alpha^{(0)} = VBV^T\varphi_\alpha,$$

where B is a diagonal matrix of order $N_\varphi \times N_\varphi$, the first $M_{de}(\alpha)$ diagonal elements of which are equal to 1 and the rest are zeros.

Step 1:

$$\varphi_{\alpha,1}^{(n)}(j) = \varphi_\alpha^{(n)}(j) + \mu_1^{(n)}[I_J(j)\varphi_\alpha^{(n)}(j) - \varphi_\alpha^{(n)}(j)], \quad 1 \leqslant j \leqslant N_\varphi,$$

where

$$I_J(j) = \begin{cases} 1, & j \in J_\varphi; \\ 0, & j \notin J_\varphi. \end{cases}$$

Step 2:

$$\varphi_{\alpha,2}^{(n)}(j) = \varphi_{\alpha,1}^{(n)}(j) + \mu_2^{(n)}[\max\{\varphi_{\min}(j), \varphi_{\alpha,1}^{(n)}(j)\} - \varphi_{\alpha,1}^{(n)}(j)].$$

Step 3:

$$\varphi_{\alpha,3}^{(n)}(j) = \varphi_{\alpha,2}^{(n)}(j) + \mu_3^{(n)}[\min\{\varphi_{\max}(j), \varphi_{\alpha,2}^{(n)}(j)\} - \varphi_{\alpha,2}^{(n)}(j)].$$

Step 4:

$$x_{\alpha,3}^{(n)} = V^T\varphi_{\alpha,3}^{(n)}.$$

Step 5:

$$x_\alpha^{(n+1)}(j) = \begin{cases} \mu_4^{(n)}x_\alpha^{(0)}(j) + (1-\mu_4^{(n)})x_{\alpha,3}^{(n)}(j), & 1 \leqslant j \leqslant M_{cT}(\alpha); \\ x_{\alpha,3}^{(n)}(j), & M_{cT}(\alpha) + 1 \leqslant j \leqslant N_\varphi. \end{cases}$$

Step 6:

$$\varphi_\alpha^{(n+1)} = Vx_\alpha^{(n+1)},$$

and then steps 1)-6) are repeated for the regularized solution $\varphi_\alpha^{(n+1)}$. When the condition $\|\varphi_\alpha^{(n+1)} - \varphi_\alpha^{(n)}\| \le \varepsilon$ is satisfied the iteration process is completed and φ_α^* is assumed as $\varphi_\alpha^{(n+1)}$. For the described algorithm of a nonlinear regularization the following statement holds: When conditions (6) are satisfied the sequence $\{\varphi_\alpha^{(n)}\}$ converges coordinates by coordinate to the limiting point φ_α^* belonging to the set $\Phi_0 = \bigcap_{i=1}^{3} \Phi_i$. In order to accelerate the convergence it is recommended that in one of the first iterations the third step of the algorithm should be replaced by the following one:

$$\varphi_{\alpha,3}^{(n)}(j) = \varphi_{\alpha,2}^{(n-1)}(j) + \mu_3^* [\varphi_{\alpha,2}^{(n)}(j) - \varphi_{\alpha,2}^{(n-1)}(j)], \quad 1 \leqslant j \leqslant N_\varphi, \tag{8}$$

where

$$\mu_3^* = \Big[\sum_{j=1}^{N_\varphi} (\varphi_{\alpha,2}^{(n-1)}(j) - \varphi_{\alpha,3}^{(n-1)}(j))^2\Big] \Big/ \Big[\sum_{j=1}^{N_\varphi} (\varphi_{\alpha,2}^{(n-1)}(j) - \varphi_{\alpha,2}^{(n)}(j))(\varphi_{\alpha,2}^{(n-1)}(j) - \varphi_{\alpha,3}^{(n-1)}(j))\Big].$$

Here $\mu_i^{(n)} = 1$, $1 \le i \le 4$. As the computational experiment has shown, such choice of $\mu_i^{(n)}$ and the use of procedure (8) reduces the number of required iterations 3-5 times and the number of iterations in total does not exceed 30.

Results of the Computational Experiment. The described algorithm of the nonlinear regularization is realized as a system of interconnected modules in the FORTRAN-IV language entering into the complex of subroutines SVDSYS for solving ill-conditioned systems of linear algebraic equations which was developed in the Institute of Theoretical and Applied Mechanics, Siberian Branch, Academy of Sciences of the USSR. We list the results of one computational experiment.

The matrix K had dimensions $N_\varphi = 10$, $N_f = 20$, and the ratio of singular numbers $\lambda_{max}/\lambda_{min} \sim 10^{10}$ (the case of the ill-conditioned matrix). We specified two similar unit projections in the vector φ (Fig. 1) and distorted the right-hand side by a noise with relative level equal to 1%. From noisy data we constructed regularized solutions: φ_α (a linear algorithm) and φ_α^* (a nonlinear algorithm with restrictions $\varphi_{min}(j) = 0$, $\varphi_{max}(j) = 2$), which are shown in Fig. 1. In the solution φ_α^* there are no negative projections observed in φ_α (the distinctive characteristic of the linear regularization algorithm in the reconstruction of δ-type solutions), and it is obvious that the accuracy of φ_α^* with closely arranged projections being allowed is higher.

In conclusion, we note that the presented algorithm of the nonlinear regularization is of high computational efficiency and can be used for solving problems on mini-computers and personal computers in automated scientific research systems.

NOTATION

K, system matrix; φ, f, vector of an unknown solution and of the right-hand side of the system of equations; N_φ, N_f, dimensions of vectors φ and f; U, Λ, and V, matrices entering into a singular expansion; λ_j, singular numbers; x_α, φ_α, vectors of a regularized solution; α, regularization parameter; $M_{de}(\alpha)$, $M_{co}(\alpha)$, number of degrees of freedom and number of constraints of the regularizing algorithm; $m(\lambda)$, stabilizing factor; σ_η^2, noise dispersion in the right-hand side; η, noise vector on the right-hand side; $r_\alpha(f)$, regularizer; Φ_i, admissible sets; J_φ, set of nonzero $\varphi(j)$; P_i, operator of projection into the set Φ_i; T_i, iteration operator; I, unit operator; μ_i, relaxation parameter; $\varphi_\alpha^{(n)}$, vector of solution for the n-th iteration.

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SOLUTION OF THE INTERNAL INVERSE PROBLEM FOR A BULK ANISOTROPIC BODY

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An approach to the problem of modeling temperature fields of bulk anisotropic bodies is developed and realized on the basis of the equivalence principle.

Composite materials and structures that are bulk bodies from a set of elements consisting from diverse materials start to be used more and more in recent years. Radio electronic apparatus (REA) that must be considered as a large system are a typical example of such structures. The thermal diagrams of the functional REA subsystems are objects requiring a systemic and hierarchical approach. These methodological modes of solving thermal problems when using a whole series of principles of the phenomenological theory of heat and mass transfer permit optimization of the thermal regimes of the REA themselves and the thermal regimes of their production.

Thus, large systems consisting of many subsystems with sharply differentiated properties and numerous heat sources are the objects of mathematical modeling, computational exeriment or methods and facilities of calculational thermophysics. The primary subsystems (homogeneous bulk bodies of complex shape), bulk and surface heat sources (sinks) produce large thermal systems for whose mathematical models of the thermal regimes there are no real values of the thermophysical characteristics, exact coordinates, and heat sources sufficiently well known in space and time.

One of the possible means of investigating the thermal regimes of such thermal systems is the experimental–theoretical approach. Underlying it are the effective (equivalent) values of the other characteristics in the condition of single-valuedness of the mathematical model (MM) of heat transport.

The equivalent (effective) values of the listed quantities can be obtained from solutions of the inverse problem [1]. Solutions of the direct problems (DP) and the optimal control problems (OCP) can be obtained by using such effective (equivalent) quantities or functions if the equivalence principle of the phenomenological theory of heat conduction is used [2]. Its crux is the following. As an example we consider the problem in which the thermophysical characteristics of an anisotropic structure will be equivalent. In substance, the mode of determining the thermophysical characteristics of an anisotropic body is elucidated when homogenization is performed. The homogenization principle has been known long and used successfully since the time when the MM of a homogeneous body is considered instead of the MM of the thermal regime of an anisotropic body.

The equivalence principle suggests just the means of a more or less exact determination of the values of the characteristics for the MM after homogenization. Let the temperature field of an anisotropic body be described by MM1. We determine the experimental temperatures T_e on a real object or on the model of an anisotropic body at certain points. The number of thermocouples and their disposition is not regulated.

The MM2 will be the model for the solution of the inverse problem (inverse, internal inverse, coefficient). This model of the thermal regime is written for a homogeneous body. We find the constant thermophysical characteristics by any method of solving the inverse

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